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1987 J. Phys. A: Math. Gen. 20 2235

(http://iopscience.iop.org/0305-4470/20/8/039)

J. Phys. A: Math Gen. 20 (1987) 2235-2237. Printed in the UK

COMMENT

A note on the perturbation expansion in the percolation problem

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Received 5 June 1986

Abstract. A new field theory representation for the percolation problem is derived by explicitly taking the $n=0$ limit in the usual $(n+1)$-state Potts model formulation. This representation is used to investigate the validity of an analytic continuation recently introduced to determine the large-order behaviour of the perturbation expansion in this problem.

Recently one of us (McKane 1986) carried out a study of the large-order behaviour of the perturbation expansion in the field theoretic version of the percolation problem, which can be formulated as the $n=0$ limit of the $(n+1)$-state Potts model.

In the course of the calculation of the large-order behaviour, a prescription for analytically continuing from the positive integers was used in order to allow the $n=0$ limit to be taken. The procedure was illustrated in detail for the zero-dimensional field theory where it was shown that, as a consequence of the continuation, the form of the Kth-order term in the perturbation expansion for large K was unlike that found in previous studies (where the number of field components has been a positive integer). The purpose of this comment is to show how this result can be obtained by a more direct evaluation. Delicate questions concerning the analytic continuation are now avoided, but the fact that the structure of the previous result is recovered is an indication of the validity of the original procedure.

Our starting point is the $(n+1)$-state Potts model expressed as a field theory

$$Z_n(g)=\int\prod_x \mathrm{d}\boldsymbol{\phi}_x \exp-\int \mathrm{d}^dx[\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i)+\tfrac{1}{2}r\phi_i\phi_i+\tfrac{1}{3}g\rho_{ijk}\phi_i\phi_j\phi_k] \qquad (1)$$

where the summation convention is assumed, $i=1,2,\ldots,n$,

$$\rho_{ijk}=\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha \qquad (2)$$

and where $\{e_i^1, e_i^2, \ldots, e_i^{n+1}\}$ is a set of $(n+1)$ n-component vectors that satisfy

$$\sum_{i=1}^{n} e_i^\alpha e_i^\beta=(n+1)\delta^{\alpha\beta}-1 \qquad \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha=(n+1)\delta_{ij} \qquad \sum_{\alpha=1}^{n+1} e_i^\alpha=0. \qquad (3)$$

We now introduce $(n+1)$ new fields ψ^α:

$$\psi^\alpha=\frac{1}{(n+1)^{1/2}}\sum_{i=1}^{n} e_i^\alpha\phi_i \qquad (4)$$

constrained by

$$\sum_{\alpha=1}^{n+1} \psi^{\alpha} = 0. \tag{5}$$

The Jacobian is equal to $(n+1)^{1/2}$ for each point in the d-dimensional space. The delta-function constraint resulting from (5) can be eliminated by the introduction of a new field, h. This leads to the following representation for $Z_n(g)$:

$$Z_n(g) = \int \prod_x \left(\frac{\mathrm{d}h_x}{2\pi}(n+1)^{1/2}\right)(\mathscr{Z}_n(g,h))^{n+1} \tag{6}$$

where

$$\mathscr{Z}_n(g,h) = \int \prod_x \mathrm{d}\psi_x \exp\left(-\int \mathrm{d}^d x[\tfrac{1}{2}(\nabla\psi)^2 + \tfrac{1}{2}r\psi^2 + (n+1)^{3/2}(g/3)\psi^3 - \mathrm{i}h\psi]\right). \tag{7}$$

The dependence on n is now explicit and the $n=0$ limit can be taken directly:

$$\lim_{n\to 0}\frac{1}{n}[Z_n(g)-1] = \int \prod_x \left(\frac{\mathrm{d}h_x}{2\pi}\right)\mathscr{Z}_0 \ln \mathscr{Z}_0 + \text{constant}. \tag{8}$$

We now wish to compute the imaginary part of the zero-dimensional version of (8), generated when the coupling constant g is pure imaginary, in order to compare with the result obtained by McKane (1986). The zero-dimensional version of (1) is

$$Z_n(g) = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\phi} \exp[-(\tfrac{1}{2}\phi_i\phi_i + \tfrac{1}{3}g\rho_{ijk}\phi_i\phi_j\phi_k)] \tag{9}$$

where for convenience the ϕ_i are scaled so that $r=1$. This integral exists and is real for g pure imaginary and for definiteness we set $g = \mathrm{i}|g|$. Then (8) can be written, dropping the subscript 0, as

$$\lim_{n\to 0}\frac{1}{n}[Z_n(g)-1] = \int_{-\infty}^{\infty}\frac{\mathrm{d}h}{2\pi}\mathscr{Z}(h,g)\ln\mathscr{Z}(h,g) + \tfrac{1}{2} \tag{10}$$

where

$$\mathscr{Z}(h,g) = \int_{-\infty}^{\infty} \mathrm{d}\psi \exp - [\tfrac{1}{2}\psi^2 + \mathrm{i}(|g|/3)\psi^3 - \mathrm{i}h\psi]. \tag{11}$$

The function in (11) is bounded and real. To evaluate it in terms of known special functions we shift ϕ by $\mathrm{i}/2|g|$ to obtain (Abramowitz and Stegun 1965)

$$\mathscr{Z}(h,g) = \exp(-1/24|g|^2)\exp(-H/2|g|)(2\pi)|g|^{-1/3}\mathrm{Ai}(-H|g|^{-1/3}) \tag{12}$$

where $H = h - 1/4|g|$ and Ai is the Airy function. For $H<0$ Ai is positive and decays fast enough so that the integral in (10) exists. For $H>0$, Ai is oscillatory and where $\mathscr{Z}<0$ an imaginary part is generated in (10). Thus

$$\lim_{n\to 0}\frac{1}{n}\operatorname{Im} Z_n(\mathrm{i}|g|) = \pm\pi\exp\left(-\frac{1}{24|g|^2}\right)|g|^{-1/3}\int_0^{\infty}\mathrm{d}H\exp(-H/2|g|)F(H|g|^{-1/3}) \tag{13}$$

where

$$F(H|g|^{-1/3}) = \begin{cases}\mathrm{Ai}(-H|g|^{-1/3}) & \text{if } \mathrm{Ai}(-H|g|^{-1/3})<0\\ 0 & \text{if } \mathrm{Ai}(-H|g|^{-1/3})>0.\end{cases} \tag{14}$$

We wish to evaluate (13) for small $|g|$ in order to calculate the Kth-order term in the perturbation expansion for large K. To do this, break up the region of integration into the regions $(0, a_1)$, $(a_1, a_2), \ldots$ where the a are the zeros of the Airy function. Integration by parts then gives

$$\lim_{n\to 0}\frac{1}{n}\operatorname{Im} Z_n(\mathrm{i}|g|) = \pm 4\pi\,\mathrm{Ai}'(-a_1)|g|^{4/3}\exp\left(-\frac{1}{24|g|^2}-\frac{a_1}{2|g|^{2/3}}\right)[1+\mathrm{O}(|g|^{2/3})] \tag{15}$$

up to exponentially smaller terms. This should be compared with (3.9) of McKane (1986):

$$\lim_{n\to 0}\frac{1}{n}\operatorname{Im} Z_n(\mathrm{i}|g|) = -2(9\pi)^{1/6}(2\pi)^{1/2}|g|^{4/3}\exp\left(-\frac{1}{24|g|^2}-\frac{(9\pi)^{2/3}}{8|g|^{2/3}}\right)[1+\mathrm{O}(|g|^{2/3})]. \tag{16}$$

As discussed in § 5 of that paper the subdominant term in the exponent and the prefactor in (16) are expected to be modified by higher-order contributions coming from nearly massless modes. The extent of the modification, in this zero-dimensional case at least, can be seen using the present approach. Before discussing this in more detail let us also note that the overall sign depends on whether $\arg g = \pi/2$ or $-\pi/2$, a detail which we have not concerned ourselves with here.

The previously derived result (16) can be obtained from (15) by using the asymptotic forms of $\mathrm{Ai}(-x)$ and $\mathrm{Ai}'(-x)$ for large x (Abramowitz and Stegun 1965). The function $\mathrm{Ai}(-x)$ has zeros when

$$-\tfrac{2}{3}x^{3/2}+\frac{\pi}{4}+\mathrm{O}(x^{-3/2}) = -\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots \tag{17}$$

and therefore the first zero ($x = a_1$) occurs at $x^{3/2} \approx 9\pi/8$, i.e. $a_1 = (9\pi/8)^{2/3}$. To the same order $\mathrm{Ai}'(-a_1) = (2\pi)^{-1/2}(9\pi)^{1/6}$ and (16) is recovered. Presumably the higher-order contributions coming from the nearly massless modes generate the asymptotic expressions for a_1 and $\mathrm{Ai}'(-a_1)$. Actually as far as using the results for numerical work goes, it makes little difference since

$$\begin{aligned} &(9\pi/8)^{2/3} = 2.320\ldots && a_1 = 2.338\ldots \\ &(2\pi)^{-1/2}(9\pi)^{1/6} = 0.696\ldots && \mathrm{Ai}'(-a_1) = 0.699\ldots. \end{aligned} \tag{18}$$

Thus the leading order results for a_1 and $\mathrm{Ai}'(-a_1)$ are good to better than 1%.

In summary, the effect of the analytic continuation and the nearly massless modes has been clarified by beginning from the representation (10). It would be interesting to extend these considerations to the field theory in $d = 6 - \varepsilon$ dimensions and compare the results with McKane (1986) and also to calculate the coefficient of g^K in (10) numerically, as a further check on our method.

References

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